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A DYNAMIC MIXED PROBLEM FOR A PACKET OF ELASTIC LAYERS†

Ye. V. GLUSHKOV and Ye. V. KIRILLOVA

Krasnodar

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A previously proposed method of solving scalar [1] and vector [2] integral equations with a meromorphic Fourier kernel symbol, which arise in problems of the dynamic contact of a punch with an elastic layer, is extended to the case of a multilayered packet of finite thickness. The general scheme for the solution (the search for the zeros and the poles of the symbol of the kernel, reduction to an infinite algebraic system starting from the conditions for the removability of the singularities of the Fourier transform and the regularization of the system by taking account of the nature of the singularity of the solution at the boundary of the contact area) remains on the whole as before. The main difference in the case of a multilayered packet is the increasing difficulties in constructing the symbol of the kernel and the search for its zeros and poles in the complex plane. A description of the special features in carrying out these steps is given and numerical results which show the trajectories of the motion of the poles in the complex plane for a continuous change in the elastic properties of the layers are represented, as well as the effect of the lamination on the form of the frequency dependence of the contact rigidity of the packet. © 1998 Elsevier Science Ltd. All rights reserved.

Mixed boundary-value problems for multilayered media using Fourier integral transforms with respect to the horizontal x and y coordinates reduce to integral equations of the convolution type in the unknown contact stresses

$$\mathbf{q} = \tau|_{z=0}, \quad \boldsymbol{\tau} = \{\tau_{xz}, \tau_{yz}, \sigma_z\}:$$

$$\iint\limits_{\Omega} \mathbf{k}(x-\xi, y-\eta)\mathbf{q}(\xi,\eta)d\xi d\eta = \mathbf{f}(x,y), \quad (x,y) \in \Omega \tag{1}$$

or to the equivalent functional equations

$$\mathbf{K}(\alpha_1,\alpha_2)\,\mathbf{Q}(\alpha_1,\alpha_2) = \mathbf{F}(\alpha_1,\alpha_2) + \mathbf{\Phi}(\alpha_1,\alpha_2) \tag{2}$$

in the two unknowns $\mathbf{Q}$, $\mathbf{\Phi}$ [3, 4]. Here, $\mathbf{f} = \mathbf{u}|_{z=0}$, $(x, y) \in \Omega$ is a specified displacement of the surface in the contact area Ω and $\mathbf{K}$, $\mathbf{Q}$, $\mathbf{F}$, $\mathbf{\Phi}$ are the Fourier transforms of the matrix-kernel $\mathbf{k}$ and the vectors $\mathbf{q}$, $\mathbf{f}$, $\varphi(\varphi = \mathbf{u}|_{z=0}, (x, y) \notin \Omega)$, respectively.

When the contact area Ω has an arbitrary shape, direct numerical methods are used to solve Eq. (1). These involve approximating the contact area by a grid and expanding the unknown function $\mathbf{q}$ in a certain system of basis functions (for example, see the variational-difference method‡). When Ω is a strip or a circle, Eq. (1) becomes one-dimensional (integration only with respect to ξ or with respect to $\rho = \sqrt{(\xi^2 + \eta^2)}$) and the functional relation (2) correspondingly also depends only on the single variable $\alpha = \alpha_1$ or $\sqrt{(\alpha_1^2 + \alpha_2^2)}$. Analytical approaches, like the Wiener–Hopf method, are used to solve it together with an expansion in splines or orthogonal polynomials with a weight which takes account of the nature of the singularity of the solution at the punch boundaries. If, in this case, the Fourier symbol of the kernel $\mathbf{K}$ is a meromorphic function (as, for example, in the case of a packet of layers of finite thickness), the solution can be obtained in closed form as a series with unknown coefficients s_k which are determined from the infinite algebraic system

$$\sum_{k=1}^{\infty} \mathbf{a}_{lk}\mathbf{s}_k = \mathbf{f}_l, \quad l = 1,2,3,\ldots \tag{3}$$

A method was proposed in [1, 2] for reducing the problem to system (3) and for its regularization.

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This, as in the classical Wiener–Hopf method, it does not require factorization of the symbol of the kernel **K**. This is particularly important when Eq. (2) is a vector equation (contact with adhesion, problems of the diffraction of normal modes at a horizontal crack, inclusion, interlayer defect, etc.) since the factorization of the matrix **K** is a separate and quite tedious problem.

In the case of a multilayered packet, if the poles ζ_k of the elements of the matrix **K** and the zeros z_l of its determinant (the roots of the functions $\Delta(\alpha)$, $\Delta_2(\alpha)$ in [2]) have already been found, the subsequent scheme for constructing the solution does not formally differ in any way from the scheme which has been described previously for a homogeneous layer [1, 2]. However, the elements of $\mathbf{K}(\alpha)$ (the functions M, P and R [2]) are not yet written out here in explicit form and the zeros and poles z_l, ζ_k are not set up in the complex plane α along the branches of the graph of the exponential function in accordance with the known [3] asymptotic behaviour for $l, k \to \infty$. From the point of view of numerical implementation, these differences are important since:

1. the numerical construction of the elements $\mathbf{K}(\alpha)$, for example, in accordance with the algorithms which have previously been used for multilayer media, gives a coincident (removable) subset of zeros and poles $z_l = \zeta_k$ which, in order to avoid dividing by zero, have first to be analytically eliminated when forming system (3) (see [1, formula (1.9)] and [2, formula (1.6)]);

2. the asymptotic values of z_l, ζ_k are used as the initial values for finding the "distant" roots in the complex plane.

The routes by which the above-mentioned difficulties can be overcome and the special features of the modification of the numerical scheme are indicated below by taking an axially symmetric contact problem as the example.

Consider the steady-state harmonic oscillations of a packet M of elastic layers S_i of thickness h_i which are induced by specified displacements $\mathbf{f}e^{-i\omega t}$ in a circular domain Ω: $0 \leqslant r \leqslant a$, $r = \sqrt{(x^2+y^2)}$ on its surface (a circular punch); S_i: $z_i \leqslant z \leqslant z_{i-1}$, $-\infty \leqslant x, y \leqslant \infty$; $z_i = z_{i-1} - h_i$, $i = 1, 2, \ldots, M$; $z_0 = 0$, $z_M = -h$, h is the overall thickness of the packet and λ_i, μ_i, ρ_i are the elasticity constants and density of the ith layer (Fig. 1). The conditions of rigid adhesion

$$z = z_i: \quad \mathbf{u}_i = \mathbf{u}_{i+1}, \quad \boldsymbol{\tau}_i = \boldsymbol{\tau}_{i+1}, \quad i = 1, 2, \ldots, M-1 \tag{4}$$

are specified at the boundaries of separation of the layers z_i.

The lower edge is rigidly fixed

$$z = z_M = -h: \quad \mathbf{u}_M = 0 \tag{5}$$

and the conditions of contact with the vibrating punch

$$z = 0: \quad \boldsymbol{\tau}_1 = \begin{cases} \mathbf{q}(r), & (x,y) \in \Omega \\ 0, & (x,y) \notin \Omega \end{cases}, \quad \mathbf{u}_1 = \mathbf{f}(r), \quad (x,y) \in \Omega \tag{6}$$

are imposed on the upper edge, which is stress-free.

Here, $\mathbf{u}_i(r, z) = \{u_{ir}, u_{iz}\}$, $\boldsymbol{\tau}_i(r, z) = \{\tau_{i,r,z}, \sigma_{iz}\}$ are the displacement and stress vectors in a layer S_i, $\mathbf{f}(r) = \{f_r, f_z\}$ are the specified displacements of the punch and $\mathbf{q}(r) = \{q_r, q_z\}$ are the unknown contact

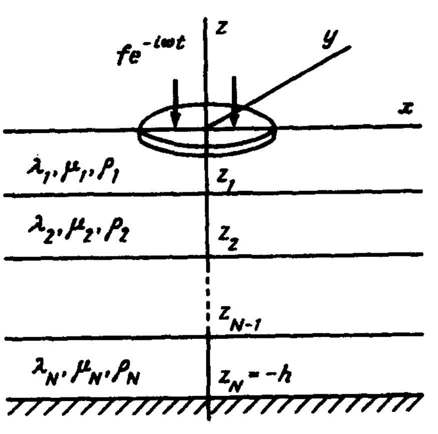

Fig. 1.

stresses in cylindrical coordinates r, φ and z (torsional vibrations are ignored). By virtue of the linearity of the problem, the harmonic factor $e^{-i\omega t}$ is omitted.

In order to construct Green's matrix $\mathbf{K}(\alpha)$, we make use of the solution in Fourier transforms of the auxiliary problem for a homogeneous layer S_i on the edges of which the loads $\mathbf{q}_i$, $\mathbf{q}_{i+1}$ are specified

$$\mathbf{U}_i(\alpha,z)=\mathbf{K}_i(\alpha,z)\mathbf{Q}_i(\alpha)+\mathbf{L}_i(\alpha,z)\mathbf{Q}_{i+1}(\alpha),$$
$$i=1,2,\dots,\ M-1 \tag{7}$$

and, in the case of the lower layer S_M with condition (5) imposed on the lower surface,

$$\mathbf{U}_M(\alpha,z)=\mathbf{K}_M(\alpha,z)\mathbf{Q}_M(\alpha) \tag{8}$$

The elements of the matrices $\mathbf{K}_i(\alpha, z)$, $\mathbf{L}_i(\alpha, z)$ are obtained in explicit form, and the system of $M-1$ vector equations

$$\begin{aligned}&\mathbf{K}_i(z_i)\mathbf{Q}_i+(\mathbf{L}_i(z_i)-\mathbf{K}_{i+1}(z_i))\mathbf{Q}_{i+1}-\mathbf{L}_{i+1}(z_i)\mathbf{Q}_{i+2}=0\\&i=1,2,\dots,M-2\\&\mathbf{K}_{M-1}(z_{M-1})\mathbf{Q}_{M-1}+(\mathbf{L}_{M-1}(z_{M-1})-\mathbf{K}_M(z_{M-1}))\mathbf{Q}_M=0\end{aligned} \tag{9}$$

in the M unknowns $\mathbf{Q}_i(\alpha)$ arises from the condition for the displacements on the boundaries of the layers (4) to be equal. This system of vector equations enables us to express $\mathbf{Q}_2, \mathbf{Q}_3, \dots, \mathbf{Q}_M$ in terms of $\mathbf{Q}=\mathbf{Q}_1$ using the recurrent matrix relations

$$\begin{aligned}&\mathbf{Q}_i=\mathbf{D}_i\mathbf{Q}_M,\quad i=1,2,\dots,M-1\\&\mathbf{D}_{M-1}=\mathbf{K}_{h,M-1}^{-1}(\mathbf{K}_{0,M}-\mathbf{L}_{h,M-1}),\quad \mathbf{D}_M=\mathbf{E}\\&\mathbf{D}_i=\mathbf{K}_{h,i}^{-1}[(\mathbf{K}_{0,i+1}-\mathbf{L}_{h,i})\mathbf{D}_{i+1}+\mathbf{L}_{0,i+1}\mathbf{D}_{i+2}]\\&i=M-2,\ M-3,\dots,1\end{aligned} \tag{10}$$

from which it follows that

$$\mathbf{U}(0)=\mathbf{K}\mathbf{Q},\quad \mathbf{K}=\mathbf{K}_{0,1}+\mathbf{L}_{0,1}\mathbf{D}_2\mathbf{D}_1^{-1} \tag{11}$$

Here, $\mathbf{K}_{0,i}$, $\mathbf{L}_{0,i}$ are the values of the matrices $\mathbf{K}_i(z)$, $\mathbf{L}_i(z)$ on the upper surface of a layer S_i: $z=z_{i-1}$, and $\mathbf{K}_{h,i}$, $\mathbf{L}_{h,i}$ are the values of the matrices on the lower surface $z=z_i$. For brevity, the dependence of the functions and the matrix-functions on α in (9)–(11) is not, as a rule, subsequently indicated.

Actually, the algorithm which is given is a version of the method of matrix propagators [6] or transfer matrices [7] for multilayer media. As in the algorithm previously used [5], the matrices $\mathbf{K}_{0,i}$, $\mathbf{L}_{0,i}$, $\mathbf{K}_{h,i}$, $\mathbf{L}_{h,i}$ do not contain increasing exponential functions but the very essential explicit form of their elements enables one to separate out and to abbreviate the factors which gives the removable zeros and poles. In particular, it follows from representation (10) in the case of a two-layer packet that

$$\mathbf{K}=\mathbf{K}_{0,1}+\mathbf{L}_{0,1}(\mathbf{K}_{0,2}-\mathbf{L}_{h,1})^{-1}\mathbf{K}_{h,1} \tag{12}$$

$$\mathbf{K}_{0,1}=\frac{\mathbf{K}_{0,1}^*}{\Delta_1},\quad \mathbf{K}_{0,1}^*=\begin{Vmatrix}i\alpha^2M_3 & -i\alpha P_3\\ i\alpha P_3 & -R_3\end{Vmatrix},\quad \mathbf{L}_{0,1}=\frac{\mathbf{L}_{0,1}^*}{\Delta_1},\quad \mathbf{L}_{0,1}^*=\begin{Vmatrix}i\alpha^2M_1 & -i\alpha P_1\\ -i\alpha P_1 & -R_1\end{Vmatrix}$$

$$\mathbf{K}_{0,2}=\frac{\mathbf{K}_{0,2}^*}{\Delta_2},\quad \mathbf{K}_{0,2}^*=\begin{Vmatrix}-i\alpha^2M_2 & -i\alpha P_2\\ -i\alpha P_2 & R_2\end{Vmatrix}$$

$$\mathbf{K}_{h,1}=\frac{\mathbf{K}_{h,1}^*}{\Delta_1},\quad \mathbf{K}_{h,1}^*=\begin{Vmatrix}-i\alpha^2M_1 & -i\alpha P_1\\ -i\alpha P_1 & R_1\end{Vmatrix},\quad \mathbf{L}_{h,1}=\frac{\mathbf{L}_{h,1}^*}{\Delta_1},\quad \mathbf{L}_{h,1}^*=\begin{Vmatrix}-i\alpha^2M_3 & -i\alpha P_3\\ -i\alpha P_3 & R_3\end{Vmatrix}$$

where

$$M_1 = \gamma_2\chi_2^2(-\gamma^2 s_1 + \alpha^2\gamma_1\gamma_2 s_2)/(2i\alpha^2),\quad P_1 = \chi_2^2\gamma_1\gamma_2\gamma(c_2 - c_1)/2$$

$$R_1 = -\gamma_1(\gamma+\alpha^2)(\gamma_1\gamma_2\alpha^2 s_1 + \gamma^2 s_2)$$

$$M_2 = \tilde\gamma_2\tilde\chi_2^2(\alpha^2\tilde c_2\tilde s_1 - \tilde\gamma_1\tilde\gamma_2\tilde c_1\tilde s_2)/(2i\alpha^2)$$

$$P_2 = (\tilde\gamma_1\tilde\gamma_2(3\alpha^2 + \tilde\gamma_2^2)\ (1 - \tilde c_1\tilde c_2) + (\alpha^2(\alpha^2 + \tilde\gamma_2^2) + 2\tilde\gamma_1^2\tilde\gamma_2^2))\ \tilde s_1\tilde s_2)$$

$$R_2 = \tilde\gamma_1\tilde\chi_2^2(\tilde\gamma_1\tilde\gamma_2\tilde c_2\tilde s_1 - \alpha^2\tilde c_1\tilde s_2)$$

$$M_3 = i\gamma_2\chi_2^2(\gamma_1\gamma_2\alpha^2 c_1 s_2 - \gamma^2 c_2 s_1)/(2\alpha^2)$$

$$P_3 = \gamma_1\gamma_2\gamma(\alpha^2+\gamma)\,(c_1c_2 - 1) - (\gamma_1^2\gamma_2^2\alpha^2 + \gamma^3)s_1s_2$$

$$R_3 = \gamma_1\chi_2^2(\gamma_1\gamma_2\alpha^2 c_2 s_1 - \gamma^2 c_1 s_2)/2$$

$$\Delta_1 = 2\mu_1(-2\alpha^2\gamma_1\gamma_2\gamma^2(1 - c_1c_2) - (\gamma^4 + \alpha^4\gamma_1^2\gamma_2^2)s_1s_2)$$

$$\Delta_2 = \mu_2((\tilde\gamma_1\tilde\gamma_2(4\alpha^2\delta - (\delta^2 + 4\alpha^4)\tilde c_1\tilde c_2) + (4\alpha^2\tilde\gamma_1^2\tilde\gamma_2^2 + \alpha^2\delta^2)\tilde s_1\tilde s_2)$$

$$\gamma_n = \sqrt{\alpha^2 - \chi_n^2},\quad n = 1,2,\quad \gamma = \alpha^2 - \chi_2^2/2$$

$$\chi_1^2 = \frac{\rho_1\omega^2}{\lambda_1 + 2\mu_1},\quad \chi_2^2 = \frac{\rho_1\omega^2}{\mu_1},\quad \tilde\chi_1^2 = \frac{\rho_2\omega^2}{\lambda_2 + 2\mu_2},\quad \tilde\chi_2^2 = \frac{\rho_2\omega^2}{\mu_2}$$

$$\delta = \alpha^2 + \tilde\gamma_2^2,\quad c_n = ch\gamma_n h_1,\quad s_n = sh\gamma_n h_1,\quad n = 1,2$$

for $\tilde\gamma$, $\tilde\gamma_n$, $\tilde c_n$, $\tilde s_n$, etc., the representations are the same with χ_n replaced by $\tilde\chi_n$ ($n = 1, 2$) and h_1 by h

$$(\mathbf{K}_{0,2} - \mathbf{L}_{h,1})^{-1} = \frac{\Delta_1\Delta_2}{\alpha^2(iP_4^2 - M_4R_4)}\mathbf{K}_4^*,\quad \mathbf{K}_4^* = \begin{Vmatrix} -iR_4 & \alpha P_4 \\ -\alpha P_4 & -\alpha^2 M_4 \end{Vmatrix} \tag{13}$$

$$M_4 = \Delta_1 M_2 - \Delta_2 M_3,\quad P_4 = \Delta_1 P_2 - \Delta_2 P_3,\quad R_4 = \Delta_1 R_2 - \Delta_2 R_3$$

The relations

$$\alpha^2(iP_4^2 - M_4R_4) = \Delta_2 S,\quad S = S^*\Delta_1$$
$$S^* = i(\Delta_1 T/\mu_2 + \Delta_2 D/\mu_1)/2 + \alpha^2(M_3R_2 + M_2R_3 - 2iP_2P_3)$$
$$D = (\alpha^4 + \gamma^2)\gamma_1\gamma_2c_1c_2 - \alpha^2(\gamma_1^2\gamma_2^2 + \gamma^2)s_1s_2 - 2\alpha^2\gamma_1\gamma_2\gamma$$
$$T = 2\alpha^2\tilde\gamma_1\tilde\gamma_2(1 - \tilde c_1\tilde c_2) + (\alpha^4 + \tilde\gamma_1^2\tilde\gamma_2^2)\tilde s1\tilde s2$$

enable us to separate out and abbreviate the factors Δ_1, Δ_2 in (13). As a result, the matrix $\mathbf{K}$ takes the form

$$\mathbf{K} = (\mathbf{K}_{0,1}^* + \mathbf{L}_{0,1}^*\mathbf{K}_4^*\mathbf{K}_{h,1}^*/S)/\Delta_1 \tag{14}$$

In the scalar case of frictionless contact, the final expression for U_z, which does not contain removable singularities, is rewritten in the following form

$$U_z(0) = K_{22}Q_z = YQ_z/(\Delta_1 S^*)$$
$$Y = -S^*R_3 + i\alpha^2 P_1(P_1R_2 - 2R_1P_2) + R_1^2 M_2 - \Delta_2 X/(2\mu_1)$$
$$X = -i\gamma_1^2\gamma_2\chi_2^4((\gamma+\alpha^2)(\alpha^2\gamma_1\gamma_2 s_1 c_1 + \gamma^2 s_2 c_2) - 2\gamma\alpha^2(\gamma c_1 s_2 + \gamma_1\gamma_2 c_2 s_1))/4$$

In the general case, the zeros z_l are found as the zeros of the determinant of the matrix $\mathbf{K}(\alpha)$ (in the scalar case, they are the zeros of the numerator of the corresponding element of $\mathbf{K}(\alpha)$) and the poles ζ_k are found as the roots of the equation $\Delta_1(\alpha)S^*(\alpha) = 0$.

It was shown in [8] that, when there is sufficient contrast in the elastic properties of the layers, the dispersion equation in the case of large α splits into the asymptotically independent dispersion equations corresponding to each layer separately. This fact can be used when choosing the initial values of the "distant" roots. However, a search based on a stepwise refinement of their position as the elastic properties of the layers are smoothly changed, beginning from the known static ($\omega = 0$) distribution for a homogeneous layer, turns out to be more reliable and also works in the case when there is a small contrast between the layers. When the required values of the elasticity moduli λ_i, μ_i and the densities ρ_i of the layers have been obtained, the resulting collection of values of z_l^0, ζ_k^0 is stored in a separate file and the subsequent scheme for searching for $z_l(\omega)$, $\zeta_k(\omega)$ in the case of a certain required frequency is the same as that used previously [1, 2]: the trajectory of the motion of the first 10–30 roots, which depend strongly on the frequency ω, is also traced in a stepwise manner in the complex plane when $0 \leqslant \omega \leqslant \omega_1$ and the remaining roots are refined by the method of parabolae, starting from the initial values z_l^0, ζ_k^0 which thereby play the role of the asymptotic form of the roots in the previous scheme. Their number must be sufficient for summation, with the required accuracy, of the series which arise in the regularization of system (3).

As an example, the trajectories of the drift of the poles ζ_k are shown in Fig. 2, while in Fig. 3 we show zeros z_l in the complex plane α, when $\omega = 0$ in the case of a two-layer packet ($h_1 = h_2 = 0.5$) and the ratio of the velocities of the transverse waves of the layers $\beta = \upsilon_{s,1}, \upsilon_{s,2}$ is changed. The initial position of the first 30 z_l, ζ_k for a homogeneous layer with the dimensionless parameters $h = 1$, $\upsilon_s = 1$, $\rho = 1$, $\nu = 0.3$ (ν is Poisson's ratio) are denoted by the open circles. The ratio β was varied: (1) from 1 to 0.02 for given fixed values for the lower layer S_2 (the superscript 0 is used for $\beta = 0.02$ in the numbering of the poles); (2) from 1 to 10 in the case of fixed properties of S_1 (the superscript ∞). The intermediate positions of the poles for $\upsilon_{s,1} = 0.5$; $\upsilon_{s,2} = 1$ are denoted by small crosses while the intermediate positions of the poles for $\upsilon_{s,1} = 0.5$; $\upsilon_{s,2} = 0.5$ are denoted by asterisks, respectively.

It should be noted with respect to the regularization of system (3) that the method used in [1, 2], which is based on taking account of the asymptotic behaviour of $\mathbf{s}_k$ when $k \to \infty$, is a special case of a more general hybrid scheme which occupies an intermediate position between expansion in orthogonal polynomials and reduction to infinite systems. This scheme, which was originally proposed for regularizing infinite systems in problems concerning composite and stepped waveguides [9], has also been found to be more effective when solving the contact problems in question. Within the framework of this scheme, the unknown $\mathbf{s}_k$ are replaced in system (3), starting from a certain number $k = N_1 + 1$, not by their asymptotic form as in [2], but are expressed in terms of N_2 unknown coefficients of the expansion of $\mathbf{q}(r)$ in orthogonal polynomials

$$\mathbf{q}(r) = \sum_{j=1}^{N_2} \mathbf{c}_j \varphi_j(r), \quad \varphi_j(r) = (a-r)^{-1/2} P_{j-1}^{(0,-1/2)}\left(\frac{r}{a}\right)$$

($P_j^{\alpha,\beta}(x)$ are Jacobi polynomials).

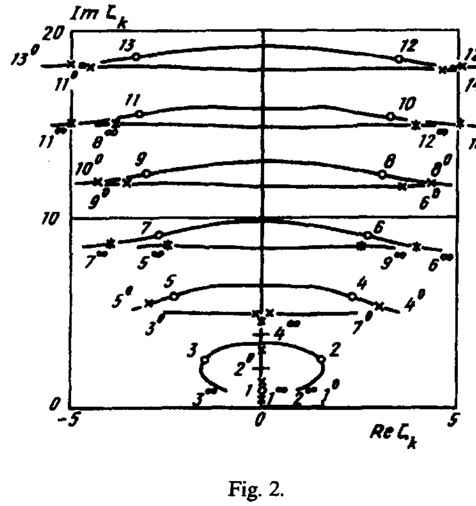

Fig. 2.

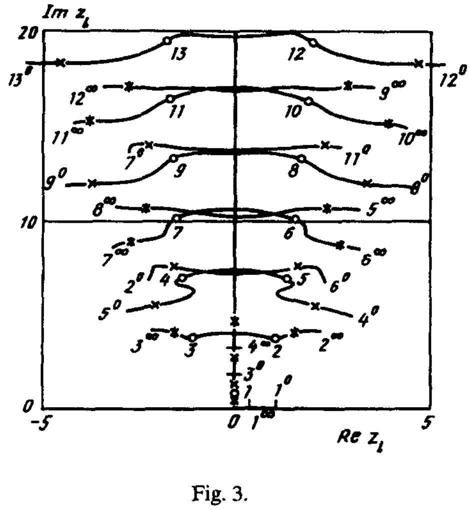

Fig. 3.

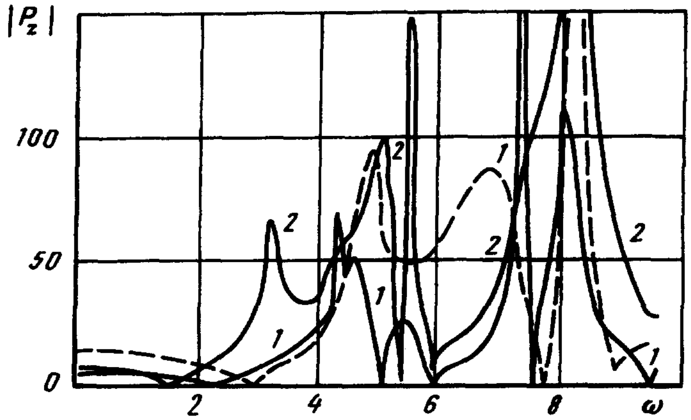

Fig. 4.

The relation between $\mathbf{s}_k$ and $\mathbf{c}_j$ follows from the same explicit representation of $\mathbf{s}_k$ in terms of $\mathbf{Q}(\zeta_k)$, which has been used previously to obtain the asymptotic form of $\mathbf{s}_k$. For example, in the case of frictionless contact [1], from the relations

$$\mathbf{s}_k = \frac{i}{2} H_0^{(1)}(a\zeta_k)\operatorname{res}\mathbf{K}(\alpha)|_{\alpha=\zeta_k}\mathbf{Q}(\zeta_k)\zeta_k,\quad \mathbf{Q}(\alpha) = 2\pi\int_0^a \mathbf{q}(r)J_0(\alpha r)rdr$$

($H_0{}^{(1)}$, J_0 are Hankel and Bessel functions), it follows that

$$\mathbf{s}_k = \sum_{j=1}^{N_2} \mathbf{d}_{kj}\mathbf{c}_j,\quad \mathbf{d}_{kj} = \pi i H_0^{(1)}(a\zeta_k)\operatorname{res}\mathbf{K}(\alpha)|_{\alpha=\zeta_k}\zeta_k\int_0^a \varphi_j(r)J_0(\zeta_k r)\,rdr$$

In this case, system (3) reduces to the well-conditioned asymptotically equivalent system

$$\sum_{k=1}^{N_1} \mathbf{a}_{lk}\mathbf{s}_k + \sum_{k=1}^{N_2} \mathbf{b}_{lj}\mathbf{c}_j = \mathbf{f}_l,\quad l = 1,2,\ldots,N;\quad \mathbf{b}_{lj} = \sum_{k=N_1+1}^{\infty} \mathbf{a}_{lk}\mathbf{d}_{kj}$$

of the comparatively small dimension $N = N_1 + N_2$. The previous method of regularization is obtained when the magnitude of N_1 is sufficiently large and $N_2 = 1$ and the value of the integral in $\mathbf{d}_{kj}$ changes its asymptotic form when $|\zeta_k| \to \infty$.

The frequency dependences of the modulus of dynamic rigidity

$$P_z = \frac{2\pi}{w}\int_0^a q(r)rdr$$

are presented in Fig. 4 as an example, where w are the vertical displacements of the rigid punch which makes frictionless contact: $\mathbf{f} = \{0, w\}$ for the two-layer packet considered above when $\beta = 0.5$ (a soft layer on a rigid layer, line 1) and $\beta = 2$ (a rigid layer on a soft layer, line 2); the radius $a = 1$. The dependence $|P_z|$ for a homogeneous layer $\beta = 1$ [1, 2] is represented by the dashed curve.

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